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Two-Equation $k-\sigma$ Turbulence Model: Application to a Supersonic Base Flow

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Base flows are an important subject of theoretical studies due to the determinant role of the rear part of projectiles, missiles, or launchers in their flight capabilities. The problem of accurately predicting the aerothermal fields in which they are embedded is still not satisfactorily solved by current theoretical methods. However, such flows are very useful test cases for improving turbulence modeling because they incorporate nearly all of the fundamental problems that can be encountered in supersonic turbulent flows submitted to strong viscous interactions. Axisymmetric configurations are appreciable for performing numerous parametrical studies of models due to their rigorous two dimensionality. In this framework, a theoretical examination of the $k-\omega$ model is performed, and its principal distinguishing characteristics with respect to $k-\varepsilon$ models are used to establish a $k-\sigma$ model whose second variable has a simple physical interpretation. A close examination of the predictive capabilities of these three types of models is done in comparison with the well documented and largely accepted database from the experiments of Herrin and Dutton (Herrin, J. L., and Dutton, J. C., "Supersonic Base Flow Experiments in the Near Wake of a Cylindrical Afterbody," AIAA Paper 93-2924, July 1993). In particular, the evolution of the fluctuating quantities predicted by the models in separated flow are given in detail. The results obtained prove that the simulation of separated flows with two-equation turbulence models is still promising in the framework of Reynolds averaged Navier–Stokes calculations.

Nomenclature

C_μ	=	constant for eddy viscosity
$e, \tilde{e}, e^*$	=	energy per unit mass; instantaneous, Favre averaged, generalized
$i, \tilde{i}, i^*$	=	internal energy; instantaneous, Favre averaged, generalized
k	=	turbulent kinetic energy
P_k	=	production term
Pr, Pr_T	=	laminar and turbulent Prandtl numbers
p, p^*	=	mean and generalized pressures
q_j^*, q_j^{*T}	=	laminar and turbulent heat fluxes
r	=	constant of perfect gases
T	=	static temperature
T_{ij}	=	total tensor of molecular constraints
t	=	time
$\tilde{U}_i$	=	mean velocity components
u, v	=	axial and radial mean velocity components
u_i''	=	fluctuating velocity components
x_i	=	Cartesian coordinates
α	=	constant of $k-\omega$ model
β	=	constant of $k-\omega$ model
γ	=	ratio of specific heats
Δ	=	Laplace operator
δ	=	constant of $k-\sigma$ model
ε	=	turbulent dissipation
η	=	constant for eddy viscosity
θ	=	temperature defined with turbulent fluctuations
λ	=	chemical potential per unit mass
μ, μ_T	=	kinematic and eddy viscosities
ν	=	kinematic viscosity coefficient
ρ	=	density
σ, σ_0	=	normalized and physical mean free paths
τ, τ_0	=	normalized and physical mean free times
τ_{ij}, τ_{ij}^T	=	molecular and turbulent shear-stress tensors
Φ_ω	=	characteristic frequency diffusive flux
χ	=	constants for diffusion terms
ω	=	frequency of crossing of eddies

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Introduction

TURBULENCE modeling with Reynolds averaged Navier–Stokes (RANS) type methods remains a useful tool for the study of complex turbulent flows whose complete direct simulation would demand prohibitive CPU times, even with the new generation of supercomputers. It has been also observed that, among RANS models, two-equation models often give better than expected results relative to their apparent simplicity. The use of methods based on the renormalization group theory (RNG) has allowed a more rigorous way of establishing the $k-\varepsilon$ model, from a mathematical point of view, but the applicability of the method is limited to incompressible cases.[1] Therefore, applications of the forthcoming evaluations of closure coefficients to compressible cases are extrapolations. The $k-\omega$ model, established with no particular assumption of incompressibility of the flow, has been satisfactorily validated in this context.[2] RNG analysis has established, by a perturbation analysis, the validity of the formula that gives traditionally the eddy viscosity of the $k-\varepsilon$ models based on dimensional arguments. Thus, the degree of confidence for the use of such formulas has been improved, and attention can be focused on the form of the source terms in transport equations that are the key elements for the calculation of eddy viscosity.

Validation of such models requires an important number of calculations to perform the adjustment phase of the constants, which is the indispensable complement of the theoretical development of the model. Such parametrical studies are still difficult to execute for three-dimensional configurations with the same present technological limitations of computers. Therefore, two-dimensional tests are still important, and among them, axisymmetric flows are the most appropriate configurations to obtain effective experimental two dimensionality.

A satisfactory prediction of the turbulence evolution in separated flows is still not assumed by any of the presently existing turbulence models. A typical configuration where an extended zone of recirculation exists is the base flow, which constitutes an excellent test case for validating any model. Base flows have been the subject of numerous experimental and theoretical studies since the 1950s to understand the physics of such flows that are of prime importance for projectiles, missiles, or space launchers. Because of the

complexity of base flows, there is still a need to constitute reliable theoretical tools for predicting the drag of afterbodies, as well as the aerothermal loads on the base region of propelled afterbodies.

The main goal of this study was to assess the ability of the k–σ two-equation turbulence model to predict the mean field and some fluctuating quantities in a supersonic base flow, where σ represents a length scale. The test case chosen is an experiment by Herrin and Dutton,[3] whose results are widely accepted for testing simulations and have served as database for previous theoretical studies on the subject.[4–7] Assessments of the model are done by comparisons with these data and with the results given by three other well-known models.

Basic Equations and Presentation of the Models

The transport equations for the mean variables of a turbulent flow, in the framework of Boussinesq's hypothesis, are written in the following form:

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}\tilde{U}_l}{\partial x_l} = 0 \tag{1}$$

$$\frac{\partial \bar{\rho}\tilde{U}_l}{\partial t} + \frac{\partial \bar{\rho}\tilde{U}_l\tilde{U}_j}{\partial x_j} = -\frac{\partial p^*}{\partial x_l} + \frac{\partial}{\partial x_j}\left(\tau_{jl} + \tau_{jl}^T\right) \tag{2}$$

$$\frac{\partial \bar{\rho}e^*}{\partial t} + \frac{\partial \bar{\rho}e^*\tilde{U}_j}{\partial x_j} = -\frac{\partial p^*\tilde{U}_j}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\left(\tau_{jl} + \tau_{jl}^T\right)\tilde{U}_l\right] - \frac{\partial}{\partial x_j}\left(q_j^* + q_j^{*T}\right) + D(k) - \left[1 - \frac{2}{3(\gamma-1)}\right](P_k - \rho\varepsilon) \tag{3}$$

with

$$D(k) = \frac{2}{3(\gamma-1)}\frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\chi_k} - \gamma\frac{\mu}{Pr} - \gamma\frac{\mu_T}{Pr_T}\right)\frac{\partial k}{\partial x_j}\right]$$

In these equations, the overbar designates Reynolds averaging and the tilde Favre averaging. The variables indexed with an * are introduced to take into account the turbulent kinetic energy. The various terms have the following definitions:

$$p^* = p + \frac{2}{3}\bar{\rho}k$$

$$p^* = (\gamma-1)\bar{\rho}i^* \qquad \text{with} \qquad i^* = \tilde{i} + \frac{2}{3(\gamma-1)}k$$

$$\tilde{i} = \frac{rT}{\gamma-1}, \qquad e^* = i^* + \frac{\tilde{U}_l\tilde{U}_l}{2}$$

$$\tau_{jl} + \tau_{jl}^T = (\mu + \mu_T)\left(\frac{\partial \tilde{U}_j}{\partial x_l} + \frac{\partial \tilde{U}_l}{\partial x_j} - \frac{2}{3}\delta_{jl}\frac{\partial \tilde{U}_m}{\partial x_m}\right)$$

$$q_j^* + q_j^{*T} = -\gamma\left(\frac{\mu}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\partial i^*}{\partial x_j}, \qquad P_k = -\overline{\rho u_k'' u_l''}\frac{\partial \tilde{U}_k}{\partial x_l}$$

where the last equation is the production term and $\rho\varepsilon$ is dissipation.

This choice of variables, considering k as a component of the pressure, keeps the familiar form of the viscous gasdynamics equations with only the exception of the two last terms of the right-hand side of the energy equation. These two terms, small in comparison with the others in this equation, are treated as source terms; the others are treated totally implicitly.[8]

The analogy between turbulent and molecular fluctuations is tied to the concept of eddy viscosity. This concept is the cornerstone of the most frequently used two-equation turbulence models, which are based on a relation between the rates of stresses and the rates of strains expressed as a first-degree polynomial function. In these models, the eddies, or local fluctuations, act essentially on the mean flow like big particles, so that the concept of eddy viscosity extends, by pure analogy, the properties of viscosity in gases from the domain of laminar flows (where it is rigorously established) to that of mixing of pseudoparticles or eddies that constitute, in this approximation, turbulent flows. Here we prefer to use the term pseudoparticle rather than eddy because a true eddy has a large scale, which is not the case in established turbulent flow, where the eddies forming very close to the wall are quickly broken. We concentrate our attention on an extrapolation to compressible flows of the k–ε RNG model,[9] on the k–ε model of Launder and Sharma (LS) (see Ref. 10), and on the k–ω model of Wilcox.[11] We will introduce a new k–σ formulation derived from the k–ω model.

The transport equations of the turbulent kinetic energy k and its dissipation rate ε in the two k–ε models are as described next.

Transport of k:

$$\frac{\partial(\rho k)}{\partial t} + \frac{\partial(\rho k\tilde{U}_i)}{\partial x_i} - \frac{\partial}{\partial x_i}\left[\left(\mu + \frac{\mu_T}{\chi_k}\right)\frac{\partial k}{\partial x_i}\right] - P_k + \rho\varepsilon + R(k) = 0 \tag{4}$$

In the case of the RNG model, $\chi_k = 0.7179$, $\mu_T = C_\mu(\rho k^2/\varepsilon)$, $C_\mu = 0.0837$, and $R(k) = 0$, and in the case of the LS model,

$$\chi_k = 1, \qquad C_\mu = 0.09, \qquad R(k) = 2\mu\frac{\partial\sqrt{k}}{\partial x_i}\frac{\partial\sqrt{k}}{\partial x_i}$$

The only term occurring in the transport of k to be modeled is the diffusion term, whose exact value is $-\frac{1}{2}\partial(\overline{\rho u_i'' u_i'' u_j''}/\partial x_j)$ because we use the following definition for ε:

$$\rho\varepsilon = -\overline{\frac{\partial T_{ij}}{\partial x_j}u_i''} \qquad \text{with} \qquad T_{ij} = -p\delta_{ij} + \tau_{ij}$$

where T_{ij} is the tensor of molecular constraints. With this definition, ε represents the work of turbulent displacements against molecular forces per unit mass and, therefore, the energy transferred from the turbulent motions to the molecular ones. The compressibility effects are absorbed by this definition of ε. To calculate compressible flows, the terms involving correlation of pressure gradients and divergence with fluctuations are often neglected. This could be considered as close to the assumption that the variable ε defined here is positive; that is, the work done by molecular forces during fluctuations is stocked as a supplementary internal energy at the molecular level.

Transport of ε:

$$\frac{\partial(\rho\varepsilon)}{\partial t} + \frac{\partial(\rho\varepsilon\tilde{U}_i)}{\partial x_i} - \frac{\partial}{\partial x_i}\left[\left(\mu + \frac{\mu_T}{\chi_\varepsilon}\right)\frac{\partial\varepsilon}{\partial x_i}\right] + C_{\varepsilon 2}\rho\frac{\varepsilon^2}{k} - C_{\varepsilon 1}\frac{\varepsilon}{k}P_k + Q(\varepsilon) = 0 \tag{5}$$

For RNG, $\chi_\varepsilon = 0.7179$,

$$C_{\varepsilon 1} = 1.42 - \frac{\eta(1 - \eta/4.38)}{1 + 0.015\eta^3}$$

$C_{\varepsilon 2} = 1.68$, $\eta = \sqrt{(P_k/C_\mu\rho\varepsilon)}$ is a function of the ratio of turbulent kinetic energy production to the dissipation rate,[9] and $Q(\varepsilon) = 0$.

For the LS model, $\chi_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92(1 - 0.3\exp -R_T^2)$, $R_T = \rho k^2/\mu\varepsilon$, and

$$Q(\varepsilon) = -2(\mu\mu_T/\rho)[(\Delta u)^2 + (\Delta v)^2]$$

The k–ω model expresses eddy viscosity as a function of the turbulent kinetic energy k and a frequency ω variously interpreted by different authors.[11] It first was considered by Kolmogorov (1942) as a specific dissipation rate and independently by Saffman (1970) as a frequency characteristic of the turbulence decay process under its self-interaction. We will keep this definition in mind in the present work. The transport equations for the k–ω model are as follows.[11]

Transport of k:

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k\tilde{U}_j) = P_k - \beta^*\rho\omega k + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\chi_k}\right)\frac{\partial k}{\partial x_j}\right] \tag{6}$$

Transport of ω:

$$\frac{\partial}{\partial t}(\rho\omega) + \frac{\partial}{\partial x_j}(\rho\omega\tilde{U}_j) = \alpha\frac{\omega}{k}P_k - \beta\rho\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\mu + \frac{\mu_T}{\chi_\omega}\right)\frac{\partial\omega}{\partial x_j}\right] \tag{7}$$

where

$$\mu_T = \alpha^*(\rho k/\omega)$$

The closure coefficients are given by

$$\alpha = \left(\frac{\beta}{\beta^*} - \frac{\chi^2}{\chi_\omega\sqrt{\beta^*}}\right)\frac{\alpha_0 + Re_t/R_\omega}{1 + Re_t/R_\omega}(\alpha^*)^{-1}$$

$$\beta^* = \frac{9}{100}\frac{5/18 + (Re_t/R_\beta)^4}{1 + (Re_t/R_\beta)^4}, \qquad \alpha^* = \frac{\alpha_0^* + Re_t/R_k}{1 + Re_t/R_k}$$

$$\alpha_0 = \frac{1}{10}, \qquad \alpha_0^* = \frac{\beta}{3}, \qquad \beta = \frac{3}{40}, \qquad \chi_k = \chi_\omega = 2$$

$$R_\beta = 8, \qquad R_k = 6, \qquad R_\omega = 2.7$$

where the turbulent Reynolds number Re_t is defined by

$$Re_t = k/\omega v$$

In the high Reynolds number limit ($Re_t \to \infty$), the quantities $\alpha \to \frac{5}{9}$, $\alpha^* \to 1$, and $\beta^* \to \frac{9}{100}$.

In the framework of the fundamental analogy leading to the Boussinesq hypothesis, a physical formula can be given for the eddy viscosity like the one established at the molecular level.[12] This formula can be written as

$$\mu_T = \eta\rho\sigma_0\sqrt{2k} \tag{8}$$

where σ_0 is a mean free path for the basic pseudoparticle of the turbulent flow in a frame moving with the mean flow, η is a constant, and $\sqrt{(2k)}$ is the local mean fluctuation velocity. The constants can be absorbed by using the variable $\sigma = \eta\sqrt{2}\sigma_0$.

If we consider now the k–ε or k–ω models with corresponding formulas for eddy viscosity, where σ is a length scale that can be evaluated by equating the definitions of this eddy viscosity, then we obtain

$$\sigma = C_\mu\left(k^{\frac{3}{2}}/\varepsilon\right) \tag{9}$$

$$\sigma = k^{\frac{1}{2}}/\omega \tag{10}$$

The method for establishing the transport equation for ε in the k–ε model begins with a rigorous derivation of the transport of fluctuations. This procedure leads to unknown terms that were evaluated by general physical or dimensional arguments in the earlier versions of the model and are now determined more rigorously in the incompressible case by RNG techniques,[13] at the cost of a considerable amount of mathematical work. In a different approach, in the k–ω model, transport for ω is directly established with physical intuition of the phenomena and dimensional arguments. In the framework of this last type of methodology, and returning to the definition of Saffman (see Ref. 11) relations (8–10) give a time $1/\omega$ equal to $\sigma/\sqrt{k}$. We can give to this time τ an interpretation based on the representation of the turbulent flow as a gas of pseudoparticles. In a frame moving with the mean flow, and by analogy with the kinetic theory of gases, we can use a formula from Ref. 12, defining the number ω_0 of pseudoparticles encountered during a unit time by one of them moving randomly at a velocity whose statistical mean is $\sqrt{(2k)}$. In our notation, this formula is

$$\omega_0 = n\varpi\sqrt{2k} \tag{11}$$

This formula supposes implicitly that ad hoc averaging has been done on the pseudoparticles of all kinds for the adaptation of the defining equations given in Refs. 12 and 14 to our present variables. Here, n is the number density of all of the pseudoparticles, and ϖ is a local mean cross section where all of the constants that appear when establishing formula (11) have been integrated. This kind of evaluation, like the fundamental hypothesis of Boussinesq, is a priori valid only when collisions between pseudoparticles are elastic, which is certainly not the case here. However, the efficiency of the subsequent models suggests that it can be partly extended to more general interactions. The number $\omega_0\tau_0$ of collisions during a time τ_0, which we call the mean free time, such that $\tau_0\sqrt{(2k)}$ is equal to the mean free path σ_0, must be equal to 1; thus, we find that ω must be equal to $1/\tau$, using the earlier defined variables ω and τ, respectively, equal to $\omega_0/2\eta$ and to $2\eta\tau_0$.

Multiplying Eq. (7) by $-1/\omega$ and adding the product $(z/k)(P_k - \rho\varepsilon)$ (the utility of which will be discussed later) to each side result in the following expression:

$$\frac{z}{k}(P_k - \rho\varepsilon) - \frac{1}{\omega}\rho\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{k}(z - \alpha)P_k + \frac{1}{k}\left(\frac{\beta}{C_\mu} - z\right)\rho\varepsilon + \frac{1}{\omega}\nabla\cdot\mathbf{\Phi}_\omega \tag{12}$$

The advection term for ω has been written in condensed form as $\rho\,\mathrm{d}\omega/\mathrm{d}t$ and the diffusion as $\nabla\cdot\mathbf{\Phi}_\omega$ where $\mathbf{\Phi}_\omega$ is the turbulent diffusive flux whose definition is

$$\mathbf{\Phi}_{\omega,j} = -\left[\left(\mu + \frac{\mu_T}{\chi_\omega}\right)\frac{\partial\omega}{\partial x_j}\right]$$

In the factor z/k, which multiplies the difference of local incoming and exiting power densities, the value of the nondimensionalized positive function z has to be determined, as well α and β. However, we will see that it is reasonable to assume the condition

$$\alpha < z < \beta/C_\mu \tag{13}$$

These power densities, integrated over a time $\mathrm{d}t$, are external work done by the mean flow and internal work done by the fluctuations.[11] We make the assumption that their difference is transmitted to the system under the form of extra fluctuations, analogous to a thermal agitation, which can be defined as a heat $\mathrm{d}Q$. The second left-hand side term can be interpreted as the rate of new interacting pseudoparticles present in the surrounding of one of them, considered as the basic material point. If we assume that the fluctuations can be considered, with a good approximation, as divergence free in a frame moving with the mean flow, this rate is assumed to be equal to the global rate $(1/n)(\mathrm{d}n/\mathrm{d}t)$. This notion is close to that of mole fraction gained or lost by the system in chemical problems. We can write Eq. (12) in a more suggestive form by defining, as in the kinetic theory of gases,[14] a temperature θ as a function of the mean kinetic energy of fluctuations per unit mass by the relation $k = \Gamma\theta$. In this relation, Γ is a constant to be defined that has the dimension of an entropy per unit mass. Equation (12) can be rescaled in the form

$$\rho\left(\frac{1}{\theta}\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\lambda}{\theta}\frac{\mathrm{d}n}{\mathrm{d}t}\right) = S \tag{14}$$

The left-hand side, in this last form, can be identified as the well-known generic formula defining the density of entropy variation of a gas,[12] and $\lambda = k/zn$ is, therefore, defined as a local mean chemical potential per particle. We can then characterize the right-hand side as being an explicit expression for this variation. This expression is a consequence of the complete definition of the local entropy per unit mass s of the open system of pseudoparticles as a function of three local physical variables. These variables are as follows.

1) E_m is the energy per unit mass transmitted to the molecular motions whose time derivative is ε; these motions are statistically not correlated with the turbulent fluctuations.

2) E_T is the energy per unit mass furnished by the mean motion whose time derivative density is P_k.

3) n_D is the number density of pseudoparticles gained or lost by turbulent diffusion, which is consequently a positive or negative component of the total number n.

With the same approximation as for the global rate of pseudoparticles, n_D can be defined by the equation

$$\rho \frac{1}{n}\frac{\mathrm{d}n_D}{\mathrm{d}t} = \frac{1}{\omega}\nabla \cdot \mathbf{\Phi}_\omega \tag{15}$$

Equation (14) can be written in the following form:

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = S = \rho \frac{1}{\theta}\frac{1}{z}\left[\left(\frac{\beta}{C_\mu} - z\right)\frac{\mathrm{d}E_m}{\mathrm{d}t} + (z-\alpha)\frac{\mathrm{d}E_T}{\mathrm{d}t}\right] + \rho\frac{\lambda}{\theta}\frac{\mathrm{d}n_D}{\mathrm{d}t} \tag{16}$$

By a clear identification, Eq. (16) gives the derivatives of s with respect to the three variables just defined. The functions α and β appearing in Eq. (16) are used, for instance, by the k–ω model and determined in a partly empirical way. With condition (13) imposed earlier to the triplet (α, β, z), S is ensured to be positive in the major part of the flow, but the openness of the local system of pseudoparticles allows possible negative values in restricted zones. The k–ω model has, thus, been included in the framework of broad physical hypotheses, which are consequences of Boussinesq's approximation. The derived entropy condition could be a useful guide for establishing new correcting factors for the source terms.

Transport Equation for the Mean Free Path σ

By conserving, in each of the two models, the primitive transport equation for k and by use of Eq. (9) or (10), we have the following generic form for transport equations for σ equivalent to the transport equation for ε or ω:

$$\frac{\partial \rho\sigma}{\partial t} + \frac{\partial \rho\sigma \tilde{U}_j}{\partial x_j} - \frac{\partial}{\partial x_j}\left|\left(\mu + \frac{\mu_T}{\chi_x}\right)\frac{\partial \sigma}{\partial x_j}\right|$$
$$= \left(\mu + \frac{\mu_T}{\chi_x}\right)\left(A\frac{\sigma}{k^2}\frac{\partial k}{\partial x_j}\frac{\partial k}{\partial x_j} + \frac{B}{\sigma}\frac{\partial \sigma}{\partial x_j}\frac{\partial \sigma}{\partial x_j} + \frac{C}{k}\frac{\partial \sigma}{\partial x_j}\frac{\partial k}{\partial x_j}\right)$$
$$+ D\frac{\sigma}{k}P_k + E\frac{\sigma}{k}\rho\varepsilon \tag{17}$$

and when the transport of σ is deduced from the transport of ε,

$$\chi_x = \chi_\varepsilon, \qquad D = \tfrac{3}{2} - C_{\varepsilon 1}, \qquad E = 1.68 - \tfrac{3}{2}$$
$$A = -\tfrac{3}{4}, \qquad B = -2, \qquad C = 3$$

When the transport of ω is used, we obtain

$$\chi_x = \chi_\omega, \qquad D = \tfrac{1}{2} - \alpha, \qquad E = \beta/C_\mu - \tfrac{1}{2}$$
$$A = \tfrac{1}{4}, \qquad B = -2, \qquad C = 1$$

This general formulation allows an easy implementation of all two-equation transport models in a single code, by passing from one model to the other with a simple exchange of the source terms constants. There is another advantage in that viscosity is a product of scalar variables. This avoids numerical divergence in regions of evanescent turbulence and rejects singularities and the use of corresponding limiters to the source terms. The k–ω calculations will be made with the k–σ variables and will be named $(k$–$\sigma)_\omega$. In this context, the wall condition for ω, which must take into account the singularity of ω at the wall,[11] will be replaced by the simpler condition $\sigma = 0$. In a purely mathematical setting, this condition is a consequence of the $1/y^2$ near-wall-like behavior of ω and of the defining relation (10). On a physical basis, it comes from the mean free time being null because the pseudoparticles are not yet individualized at the wall.

An important general result of the RNG analysis applied by Yakhot and Orszag to incompressible turbulent flows[13] is the equality, at high turbulent Reynolds numbers, of the constants connecting eddy viscosity to the turbulent diffusivities of the scalars transported in the flow. The effective constancy of this generalized Prandtl number is a fundamental hypothesis well established by experiment. Thus, in this context, a common value has been deduced for the classical Prandtl number Pr_T and the constants χ_k and χ_ε; this value is 0.7179 (see Ref. 9). This fundamental result will be extrapolated to the following compressible application for the diffusion terms in k and σ transport equations.

In Eq. (17), the product $C_{\varepsilon 1}P_k$ appearing in the first source term when the RNG is used is equal to $1.42P_k$ + (higher-order terms of the mean strain rate) in the case of weak strain rates[9]; the homologous part of this source term, when the k–ω model is used, is αP_k and is equal to $(\frac{5}{9})P_k$ in a fully turbulent flow.[11] Therefore, these corrected contributions to the first source term in fully turbulent and weakly strained flows (whose limit case is, for instance, a grid-generated turbulence in a uniform mean flow[15]) are in a ratio δ equal to $(\frac{5}{9})/1.42$. The basic idea for improving the k–σ formulation was to transition from the $(k$–$\sigma)_\omega$ model to a model $(k$–$\sigma)_1$ by replacing the contribution αP_k by a function of $C_{\varepsilon 1}P_k$, which is null for zero values of P_k. While linearly approximating this functional dependence, we fix the remaining proportion coefficient by forcing the $(k$–$\sigma)_1$ model to be equivalent to the $(k$–$\sigma)_\omega$ when the flow approaches the limit case of a homogeneous turbulence submitted to negligible mean gradients.[15] This condition is realized by substituting $\delta C_{\varepsilon 1}$ for the correcting factor α, which gives the $(k$–$\sigma)_1$ a more complex RNG-like correction in the high gradient areas of the flow. The correcting factor that, in the k equation, multiplies $\rho\varepsilon$ in the k–ω model is not used in the $(k$–$\sigma)_1$ model, and ε is replaced in the two transport equations by its value obtained from Eq. (9): $C_\mu k^{3/2}/\sigma$. Explicitly, the transport equation for σ in the $(k$–$\sigma)_1$ model is

$$\frac{\partial \rho\sigma}{\partial t} + \frac{\partial \rho\sigma \tilde{U}_j}{\partial x_j} - \frac{\partial}{\partial x_j}\left|\left(\mu + \frac{\mu_T}{0.7179}\right)\frac{\partial \sigma}{\partial x_j}\right|$$
$$= \left(\mu + \frac{\mu_T}{0.7179}\right)\left(\frac{1}{4}\frac{\sigma}{k^2}\frac{\partial k}{\partial x_j}\frac{\partial k}{\partial x_j} - \frac{2}{\sigma}\frac{\partial \sigma}{\partial x_j}\frac{\partial \sigma}{\partial x_j} + \frac{1}{k}\frac{\partial \sigma}{\partial x_j}\frac{\partial k}{\partial x_j}\right)$$
$$+ \left(\frac{1}{2} - \delta C_{\varepsilon 1}\right)\frac{\sigma}{k}P_k + \left(\frac{\beta}{C_\mu} - \frac{1}{2}\right)\rho C_\mu k^{\frac{1}{2}} \tag{18}$$

The first performed parametrical studies have shown that modifications of α and β such that condition (13) is violated lead to divergences of the calculation or drastically unphysical solutions. It seemed interesting to perform a parametrical study by perturbing slightly the basic $(k$–$\sigma)_1$ model, to test its sensitivity to variations of the production source term controlled by the parameter δ. Several calculations were performed with changes in the values of δ, which modified the results without improvement. The variant of the k–σ model corresponding to the value $(\frac{6}{9})/1.42$, in which the results are close to those given by the $(k$–$\sigma)_\omega$ model, was retained as an example and named $(k$–$\sigma)_2$. The important effect of this empirical matching of a correcting source term will illustrate that the evaluation of the creation/destruction term is a key factor for the determination of the Reynolds stresses.

Application of the Models to the Prediction of a Base Flow

Computed Case and Conditions of Calculations

The described models will be validated on data provided by the experiments of Herrin and Dutton,[3] which were performed in a test facility specially designed to generate axisymmetric flows.[16] In particular, profiles of mean and fluctuating velocity fields in the recirculating flow are furnished by laser Doppler velocimetry (LDV) measurements. The real Mach number has been determined from LDV[3] to be $2.46 \pm 1\%$, whereas the static wall pressure measured just upstream of the base corner corresponds to a uniform flow with a Mach number of 2.44. During the present calculations we have chosen an upstream Mach number $M_0 = 2.45$.

The present calculations have been performed by using the NASCA research code, which uses an extension of the Osher and Chakravarthy scheme[17] to accommodate locally nonuniform grids. This code has been previously validated on laminar high Mach number flows to avoid uncertainties due to turbulence models.[18] The near-base part of the rectangular 201×281 mesh and an example of calculated streamline pattern obtained with the $(k$–$\sigma)_1$ model are both shown in Fig. 1. The rapid variations of the fields and the turbulent mixing in the shear layer forming at the base shoulder are the

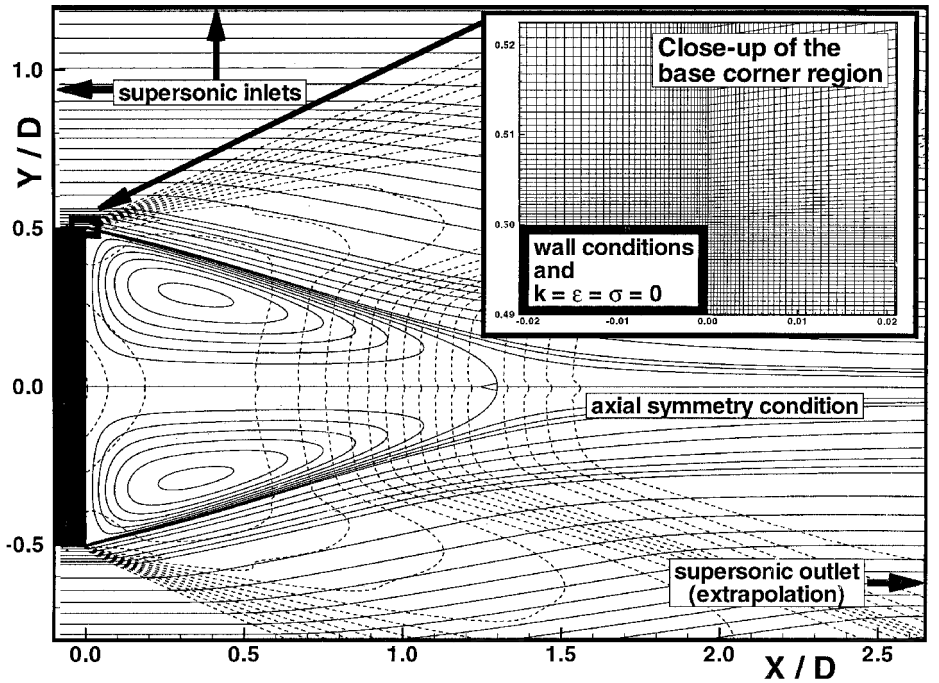

Fig. 1 Calculation grid and organization of the flow: ——, streamlines and – – –, constant pressure levels.

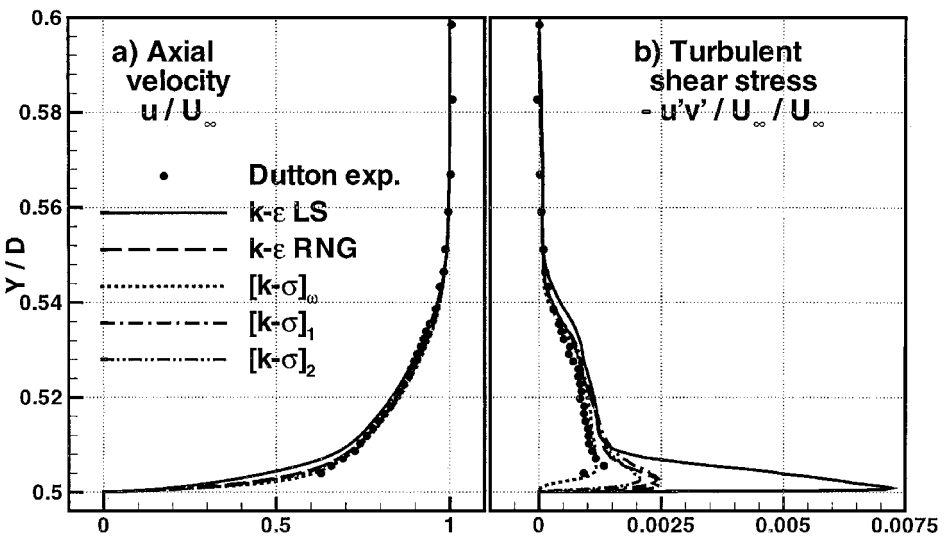

Fig. 2 Boundary-layer profiles at $X/D = -0.0157$.

first challenge for the modeling of the mean and fluctuating fields. Good precision in the prediction of the nearly constant pressure in the recirculating bubble, limited by the mixing layer and the rear stagnation point on the axis, is fundamental for base drag prediction (constant pressure levels are shown as dashed lines in Fig. 1). The recompression subsequent to the flowfield realignment at the rear stagnation point is visible in Fig. 1, at the point where the recirculating flow progressively changes into a wake. The mesh has been defined by careful study of the spatial convergence in which the optimum refining at the walls was determined, especially the transverse grid distribution on the base. The criterion for spatial convergence was to stabilize the solutions obtained with different grid resolutions within a 1% margin for the more sensitive variables, which were the pressure and the shear stress in the reversed flow region. The numerical study with each turbulence model was performed on this mesh with typical CPU times of 20–30 min on the ONERA NEC SX5 computer.

The experiment of Herrin and Dutton[3] provides us with data on the evolution of the boundary layer on a restricted part of the body beginning at a distance $X = -0.079D$ from the base (D is the cylindrical body diameter). The experimental profiles at this location are taken as the upstream boundary condition for the computations. Profiles of ε or σ are calculated on this upstream boundary from the experimental values of k and $\overline{u'v'}$ by making the theoretical evaluation of the correlation used in the general formulation equal to the measured value. The resulting computed boundary-layer profiles and cross correlation are shown in Fig. 2, just before the separation point (at $X = -0.0157D$ upstream of the base corner). At this location, the experimental profile of k is not given. Discrepancies appear in the prediction of the near-wall values of the fluctuating fields. The $(k-\sigma)_\omega$ model includes low Reynolds number corrections via the factor α and an intermittency factor α^* [see Eq. (7)]. The LS model is also equipped with such corrections [see Eqs. (4) and (5)], but the strong overprediction observed in Fig. 2 for the shear stress level in the boundary layer suggests that the diffusion and source terms used in this model, which introduce supplementary effects of high variations of k [with $R(k)$], and of the concavity of the velocity profiles [with $Q(\varepsilon)$] have to be reexamined. The following results will confirm that the estimations of the fluctuating quantities by the LS model are excessive for the majority of the flow. The RNG

model possesses in its structure the accounting of laminar–turbulent transition, avoiding in principle the use of near-wall corrections.[9,19] In fact, a recent paper suggests supplementary corrections with intermittence factors.[20] For the calculation done here, such a factor has been defined for the viscosity, in the case of the RNG and k–σ models, following theoretical arguments on the asymptotic behavior of the near-wall velocity field.[21] This factor has the following well-known general form:

$$f = 1 - \exp(-At/\tau)$$

where A is an empirical constant and t/τ has been chosen here to be the ratio of turbulent and laminar mean free times evaluated by using the considerations given in the second section. Explicitly, we have $t/\tau = (\mu_T/\mu)(rT/k)$. The local molecular mean velocity fluctuation, furnished by the kinetic theory of gases, is equal to $\sqrt{(rT)}$. The main problem for evaluating such a correction is the lack of data very close to the wall. It has also been noticed in recent experimental studies that the influence of wall roughness on Reynolds stress is considerable.[22] It has been verified that the damping factor used here avoided unrealistic levels of the computed near-wall fluctuating fields that had previously been observed in computations using uncorrected formulas.

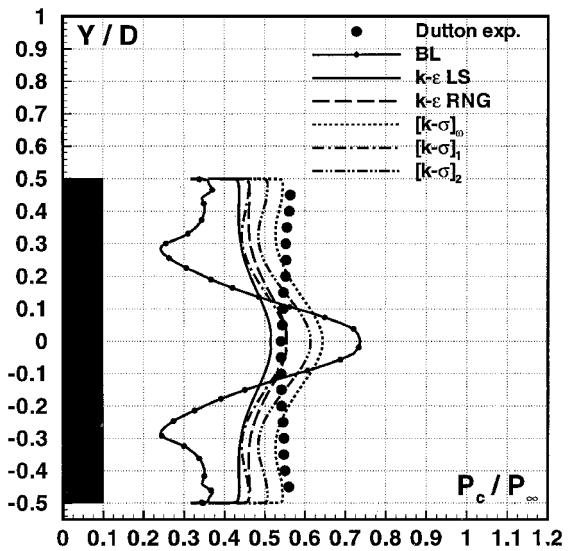

Fig. 3 Base pressure profiles.

Discussion of Results

Base Pressure Prediction

The distributions of the calculated and experimental pressure distributions on the base are shown in Fig. 3. The experimental value of the ratio p_c/p_∞ of the mean base pressure to the uniform upstream pressure is equal to 0.55. This level is 10% higher than data on base flows at the same Mach number extracted from earlier experimental compilations.[23] A satisfactory base pressure prediction is given by the RNG and k–σ models only in a region close to the axis. The values obtained in the external part of the base are underpredicted by 15–20% with the $(k$–$\sigma)_1$ model and by 25% with the LS model. This is the consequence of an important overestimation of the magnitude of the radial stream in the near-base part of the bubble. The exaggerated level of the calculated longitudinal negative velocities on the axis, decelerating while approaching the base, has induced this last effect by generating the pressure bump observed in the center of the base. A global overprediction of the expansion after separation

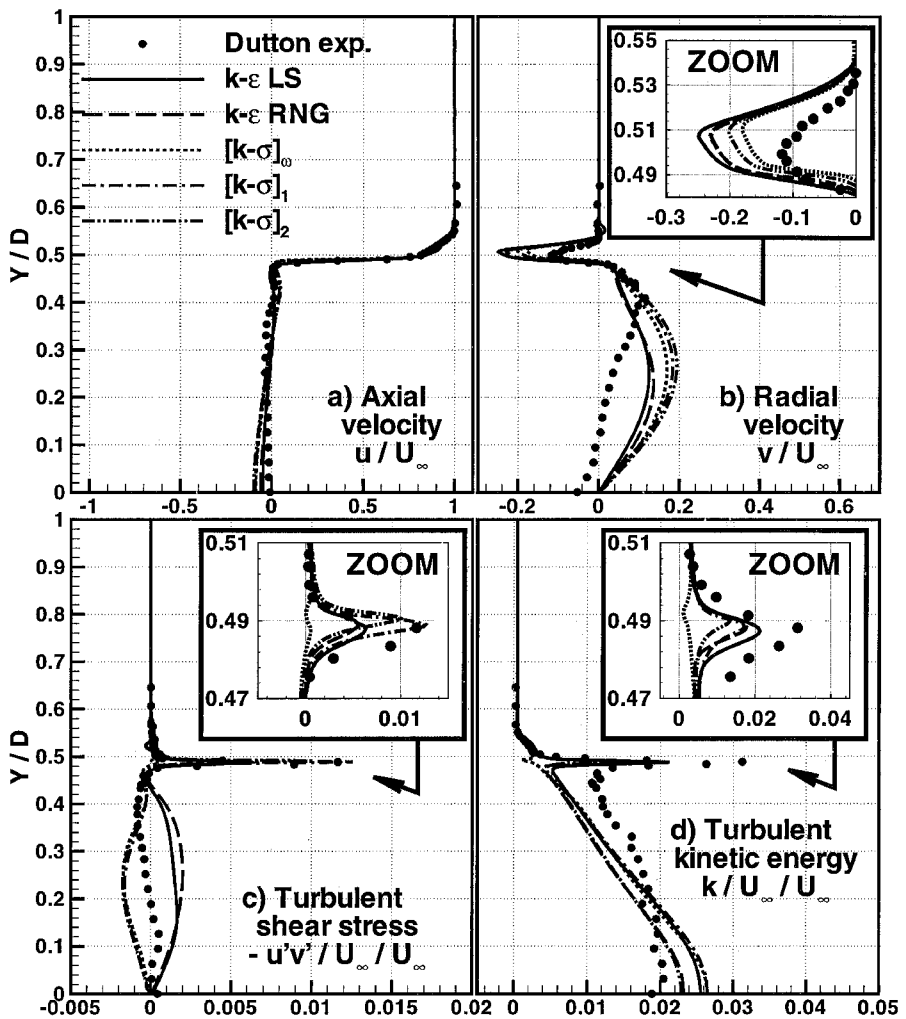

Fig. 4 Profiles at $X/D = 0.039$.

at the base shoulder characterizes all of the models. The significantly too low p_c/p_∞ ratio given by the RNG and $(k-\sigma)_1$ models are, respectively, equal to 0.49 and 0.48. The $(k-\sigma)_\omega$ model presents qualitatively the same defects, but predicts a more satisfactory mean pressure ratio of 0.52, whereas a value of 0.5 has been obtained with the $(k-\sigma)_2$ model. The pressure levels given by the LS model are too low everywhere, with the resulting mean ratio 0.46. The damping factor correcting the viscosity in the calculations is removed just downstream of the base, except for the LS and $k-\omega$ models, as recommended for their general applications in Refs. 10 and 11. These base pressure values are considerably closer to experiment than that obtained with a Baldwin–Lomax (BL) calculation (Fig. 3), in agreement with previous studies.[4] However, they contradict overpredictive results obtained with other two-equation models.[6] In a recent computation using large-eddy simulation (LES),[7] a level of 0.52 for the mean base pressure was obtained; however, a close examination of the radial profiles in the reversed flow region was not within the scope of the study. Here, a global validation considering all of the measured fields has been taken into account before focusing on base pressure values.

Radial Distributions of Field Properties

At a distance $X = 0.039D$ downstream of the base, the five models give satisfactory results in the prediction of the axial mean velocity profiles in the mixing layer (Fig. 4, with zoomed parts for the details of this layer), but radial mean velocity levels are too high. The $(k-\sigma)_1$ model gives good predictions of the shear stresses in this high-gradient zone. The LS and $(k-\sigma)_\omega$ models underestimate both the turbulent kinetic energy and shear stress equally. The quasi-zero values found by the $(k-\sigma)_\omega$ are particularly unphysical. Less satisfactory results are given by the $(k-\sigma)_1$ for the kinetic energy, similar to those given by the $k-\varepsilon$ models. The evolution between the mixing layer and the axis, of the cross correlations, is more realistic with the $(k-\sigma)_\omega$ and $k-\sigma$ models. The exaggerated level of radial velocities in the near-base part of the flow also clearly appears. This discrepancy is smaller with the $k-\varepsilon$ models. The axial velocity profiles are all satisfactory.

Profiles obtained in the middle of the bubble, at $X = 0.63D$ downstream of the base, are shown in Fig. 5. The velocity profiles in the mixing layer are well predicted by the $(k-\sigma)_\omega$ model. The evolution of k is qualitatively similar to the one given by the other models with a globally lower level, which is satisfactory on the axis but unrealistic in the mixing layer. A more accurate prediction of the shear stress characterizes both the $k-\varepsilon$ and $(k-\sigma)_1$ models. The coincidence between the $(k-\sigma)_1$ and RNG models, on the one hand, and between the $(k-\sigma)_2$ and $(k-\sigma)_\omega$ models, on the other hand, appears clearly on the k and $\overline{u'v'}$ profiles. With the two $k-\varepsilon$ models, nonrealistic positive values of the Reynolds stress appear in the expansion fan emanating from the base corner. A bump in turbulent kinetic energy profiles is also produced at the upper boundary of the mixing layer. The level of these nonexisting fluctuations are particularly important in the LS model calculation. Prediction of the maximum shear stress levels by the LS model in the mixing layer is better, but is accompanied by higher levels of kinetic energy than with the RNG and $k-\sigma$ models. A salient fact is the smoothing by the models of the shear stress evolution during the rapid transition between the mixing layer and the reversed flow. This smoothing explains the inability of the models to predict the nearly constant reversed flow

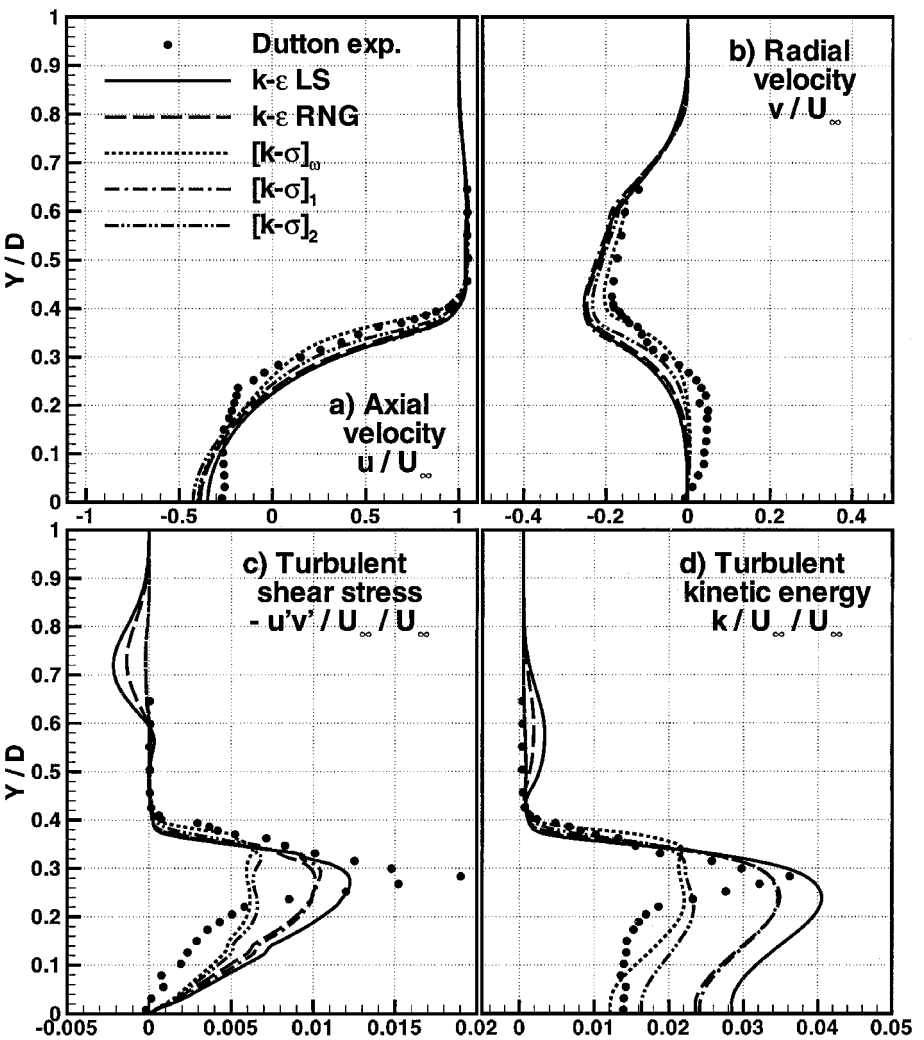

Fig. 5 Profiles at $X/D = 0.63$.

region in the immediate proximity of the base. The defect comes from an overprediction of the negative axial velocity on the axis. This too large radial variation of the axial mean velocity is also due to an excessive evaluation of the eddy viscosity in the part of the bubble situated near the axis.

The results obtained by the RNG and $(k-\sigma)_1$ models are particularly satisfactory at the rear stagnation point, located at $X = 1.26D$ (Fig. 6), except for the k profile. The good agreement of the axial velocity profile with experiment proves that the position of this stagnation point is well predicted by the two models. This result is complemented by a correct calculation of $\overline{u'v'}$. Note that the $(k-\sigma)_\omega$ and $(k-\sigma)_2$ models overestimate the separated bubble length, which is apparent from the still negative values in the axial velocity profile. This deficiency is also observed in the calculation of the Reynolds stresses; however, the profile of k obtained with these models is still rather satisfactory. Pathological behaviors of the $k-\varepsilon$ models are confirmed in shear stress and kinetic energy profiles at the level of the compression and expansion fans. The LS model significantly overpredicts the turbulent quantities. A local maximum of radial velocity is generally predicted at a distances from the axis of approximately $0.45D$. The experimental bump is situated below $0.4D$. This may be a consequence of an anticipation of the calculated position of the origin of the compression fan with respect to experiment. This compression has more effect on the transverse flow.

Farther downstream, at $X = 2.52D$, a classical wake is constituted (Fig. 7). The RNG and $(k-\sigma)_1$ models give still close results, except for the nonexisting fluctuations found by the first model in the external part of the wake. Wake velocity profiles are correct. The remaining discrepancies are an overprediction of k on the axis and shifts in the calculated position of the maximum shear stress with respect to the experiment and in the localization of the shock emanating from the stagnation point (visible as a discontinuity on the radial velocity profiles). The origin of this last discrepancy has been explained at the end of the preceding paragraph. Other discrepancies characterize the $(k-\sigma)_\omega$ model: The axial velocity and shear stress profiles are not well predicted. The behavior of the $(k-\sigma)_2$ model remains closer to that of the $(k-\sigma)_\omega$. The results of the LS model for the mean velocities fit those of the $(k-\sigma)_1$ and RNG models, but the crossing of the radial velocity profile by the shock is smoothed. The prediction of the shear-stress profile is still characterized by an unphysical bump at the level of the shock issued from the reattachment region. The important excess of turbulent kinetic energy given by this model tends to propagate to the entire wake.

The evolution of the turbulent kinetic energy on the axis is more satisfactorily predicted by the $(k-\sigma)_\omega$ and $(k-\sigma)_2$ models. However, the earlier comparisons show that this result does not constitute a sufficient criterion for establishing the global validity of the model. In fact, except for the overprediction of the reversed flow velocity along the axis, the $(k-\sigma)_\omega$ and $(k-\sigma)_2$ models on one side and the $(k-\sigma)_1$ and RNG models on the other side present complementary discrepancies. The levels of velocity fluctuations given by the LS model surpass those given by all of the other models. Note the key role of the function $C_{\varepsilon 1}$, chosen as correcting coefficient to the production source term in the $k-\sigma$ models and extracted from the RNG model. Slight modifications of the constant multiplying $C_{\varepsilon 1}$ nearly allow an interpolation between the nonequivalent RNG and $(k-\sigma)_\omega$ models.

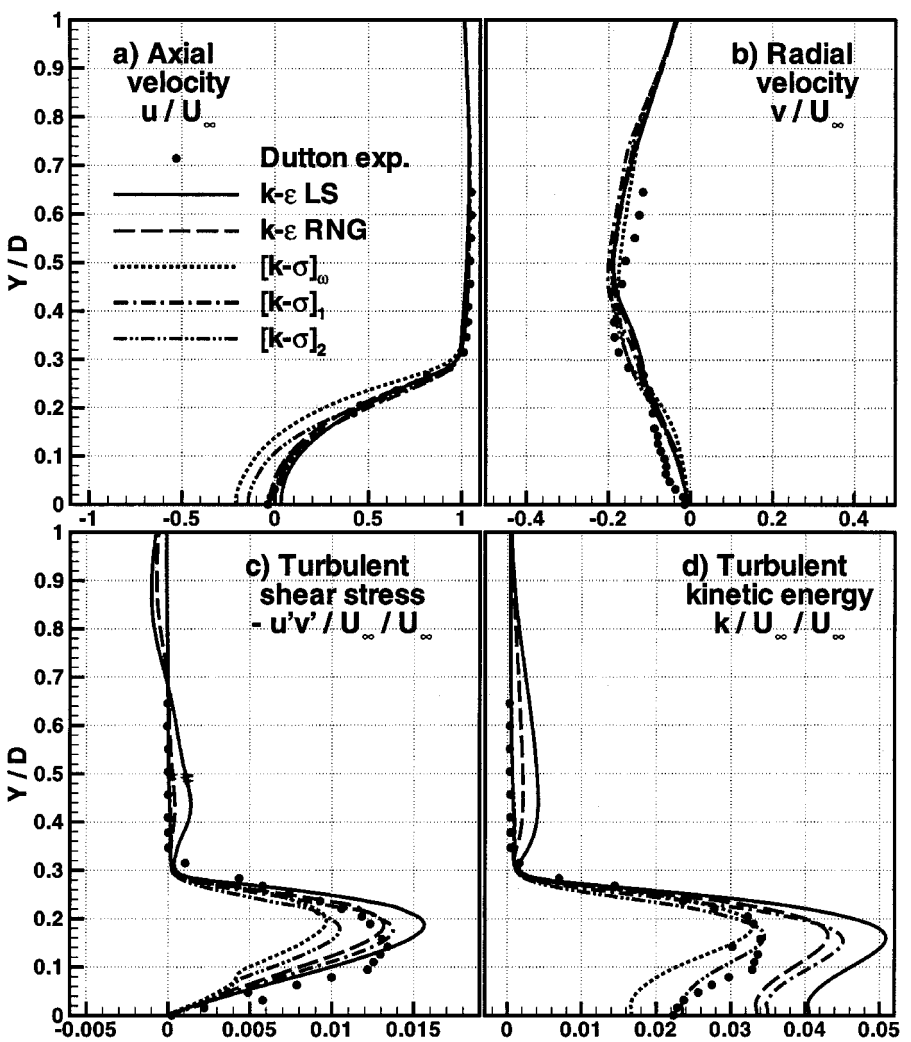

Fig. 6 Profiles at $X/D = 1.26$.

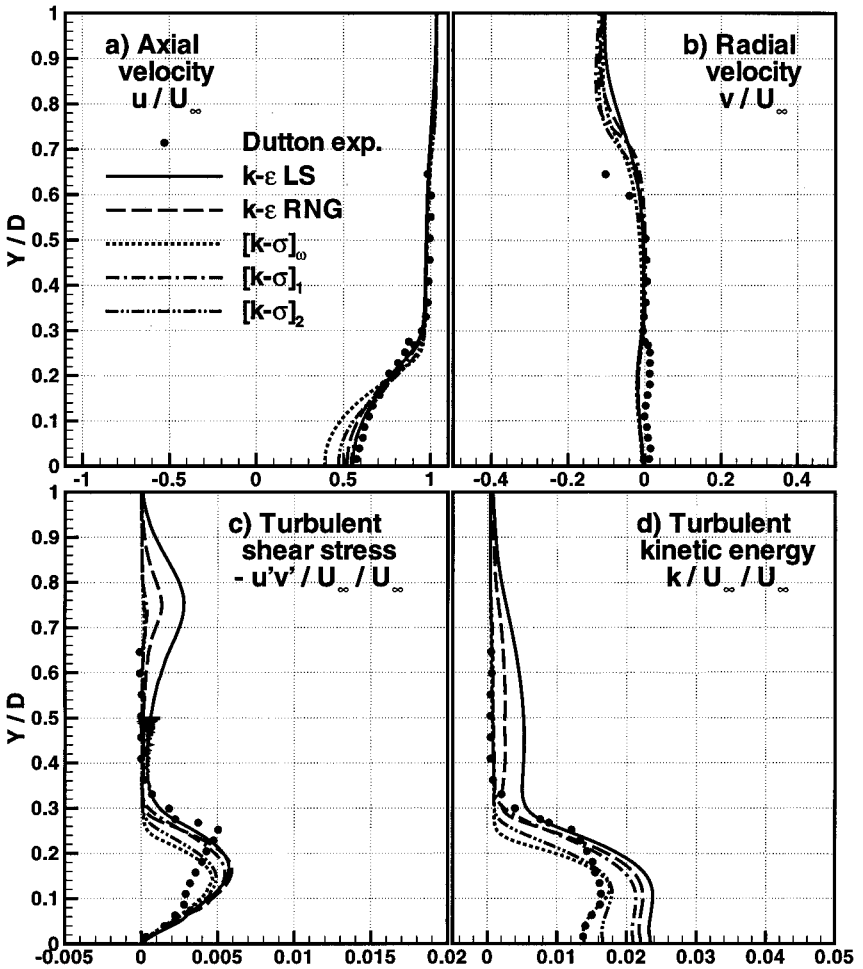

Fig. 7 Profiles at $X/D = 2.52$.

Conclusions

The starting point of the present work was an analysis of two updated two-equation turbulence models: a version of the k–ε model corrected by Yakhot and Orszag[13] by the use of RNG and the k–ω model of Wilcox.[11] The Launder and Sharma version[10] of the k–ε model was also considered for comparative evaluations. These three models are derived by methods that can be distinguished by different mathematical and physical considerations. In particular, RNG techniques are a powerful tool currently used in many branches in physics. The model used here is a special adaptation to turbulent transport in aerodynamics of these techniques.

A new physical interpretation has been established for the k–ω model with the goal of obtaining a coherence between this model and the global Boussinesq hypothesis. The result was a firmer physical justification for the ω transport equation, which was previously established on the grounds of purely dimensional arguments. The second transport equations of the k–ε and the k–ω models were rewritten using the variable σ interpreted as a mean free path for the pseudoparticles materializing turbulent fluctuations. A clearer understanding of the physical significance of the variable is gained. A key issue of this new formulation was to confirm the role of the source terms in the transport equation of the second variable for obtaining satisfactory predictions of the calculated fields. To this end, an influencing source term was localized and, after establishing a theoretically equivalent model of the k–ω, that is, $(k$–$\sigma)_\omega$, two supplementary variants, $(k$–$\sigma)_1$ and $(k$–$\sigma)_2$, were defined.

After testing the models with the specified test case, it is shown that the $(k$–$\sigma)_1$ and the RNG models provide similar results. These models appear to be more predictive than the $(k$–$\sigma)_\omega$ and the $(k$–$\sigma)_2$ models in the extended viscous zones characterizing this flow. These last two models also provide similar results. The LS model poses more serious problems due to the low overall corrective capabilities of its added source terms. This study proves that two-equation turbulence models, if not sufficiently predictive in separated flows, are still perfectible. These models will remain the most widely used models in industrial applications, when the still prohibitive cost of more complex techniques in terms of computer time are considered.

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